

Generalized Twin Prime Formulas

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Abstract

Based on Golomb's arithmetic formulas, Dirichlet series for two classes of twin primes are constructed and related to the roots of the Riemann zeta function in the critical strip.

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1 Introduction

Golomb used his arithmetic formula [1]

$$2\Lambda(2a-1)\Lambda(2a+1) = \sum_{d|4a^2-1} \mu(d) \log^2 d, \quad a \geq 2, \quad (1)$$

where $\Lambda(n)$ is the von Mangoldt function and $\mu(n)$ the Möbius function [2], [3], as coefficients of a power series which naturally converts to a Lambert series. Due to the lack of an Abelian theorem for the latter, no further progress toward a solution of the twin prime problem was possible. However, the formula may be used to construct twin prime Dirichlet series. Such a method is applied here to twin primes $p, p' = p + 2D$ for odd $D > 0$ first and then even $D > 0$. Based on the corresponding Golomb identity

$$2\Lambda(2a-D)\Lambda(2a+D) = \sum_{d|4a^2-D^2} \mu(d) \log^2 d, \quad a \geq a_D, \quad (2)$$

for the generalized twin prime numbers $p = 2a - D, p' = 2a + D$. The numbers $a_D > 0$ characterize the first twins that are a distance $2D$ apart. While the running median $2a$ between $p = 2a - D$ and $p' = 2a + D$ for odd D is an even number independent of D , whereas a_D depends on D and is more irregular. For example, $a_D = 2$ for $D = 1$; $a_D = 4$ for $D = 3$; $a_D = 4$ for $D = 5$; $a_D = 5$ for $D = 7$; $a_D = 7$ for $D = 9$, etc.

For most even D , such as $2, 4, 8, 10, 14, 16, \dots$ the median $3(2a - 1)$ between the twin primes $p = 3(2a - 1) - D$ and $p' = p = 3(2a - 1) + D$ that are a distance $2D$ apart is again a linear function of the running natural number a . Golomb's identity for these cases is

$$2\Lambda(3(2a - 1) - D)\Lambda(3(2a - 1) + D) = \sum_{d|9(2a-1)^2-D^2} \mu(d) \log^2 d, \quad a \geq a_D. \quad (3)$$

For $6|D$ the median is more irregular. Therefore, these twin prime numbers and those not of the form $(2a - D, 2a + D); (3(2a - 1) - D, 3(2a - 1) + D)$ will not be considered here.

The strategy will be to decompose the relevant Golomb identity into two factors whose generating Dirichlet series, one acting as the prime number sieve and the other to implement the constraint $n = 4a^2 - D^2$ in $\sum_{d|n} \mu(d) \log^2 d$ for the twins for odd D and $n = 9(2a - 1)^2 - D^2$ for even D , are then used in a product formula for Dirichlet series to construct the twin prime Dirichlet series.

2 Twin Prime Generating Dirichlet Series

Definition 2.1. The generating Dirichlet series of one factor of the term on the rhs of Golomb's identity (1) is defined as

$$Z(s) \equiv \sum_{n=2}^{\infty} \frac{1}{n^s} \sum_{d|n} \mu(d) \log^2 d, \quad \Re(s) \equiv \sigma > 1. \quad (4)$$

The series converges absolutely for $\sigma > 1$. Multiplying termwise the Dirichlet series

$$\frac{d^2}{ds^2} \frac{1}{\zeta(s)} = \sum_{n=2}^{\infty} \frac{\mu(n) \log^2 n}{n^s} \quad (5)$$

and $\zeta(s)$, which is justified by absolute convergence of both series for $\sigma > 1$, gives Eq. (4). Carrying out the differentiations yields

$$Z(s) = \zeta(s) \frac{d^2}{ds^2} \frac{1}{\zeta(s)} = 2 \left(\frac{\zeta'(s)}{\zeta(s)} \right)^2 - \frac{\zeta''(s)}{\zeta(s)} = -\frac{d}{ds} \frac{\zeta'}{\zeta}(s) + \left(\frac{\zeta'(s)}{\zeta(s)} \right)^2. \quad (6)$$

Among the poles of the meromorphic function $Z(s)$ are the roots ρ of the Riemann zeta function in the critical strip $0 < \sigma < 1$, which is clear from Eq. (6). The next two lemmas display analytic properties of $Z(s)$ that are needed in Section 6.

Lemma 2.1. *A pole expansion of $Z(s)$ is given by*

$$\begin{aligned} Z(s) = & \frac{-1}{(s-1)^2} + \sum_{\rho} \frac{1}{(s-\rho)^2} + \frac{1}{2} \frac{d}{ds} \frac{\Gamma'(\frac{s}{2}+1)}{\Gamma(\frac{s}{2}+1)} \\ & + \left(1 + \frac{\gamma}{2} - \log 2\pi + \frac{1}{s-1} + \frac{1}{2} \frac{\Gamma'(\frac{s}{2}+1)}{\Gamma(\frac{s}{2}+1)} - \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) \right)^2 \end{aligned} \quad (7)$$

with $\gamma = 0.57721566 \dots$ the Euler-Mascheroni constant and ρ denoting the zeros of $\zeta(s)$ in the critical strip $0 < \sigma < 1$.

Proof. The pole expansions [2],[4] of the meromorphic functions $\frac{\Gamma'(s)}{\Gamma(s)}$, $\frac{\zeta'(s)}{\zeta(s)}$,

$$\begin{aligned} \frac{\Gamma'(s)}{\Gamma(s)} &= -\gamma + \sum_{n=1}^{\infty} \frac{s-1}{n(n+s-1)}, \\ \frac{\zeta'(s)}{\zeta(s)} &= \log 2\pi - 1 - \frac{\gamma}{2} - \frac{1}{s-1} - \frac{1}{2} \frac{\Gamma'(\frac{s}{2}+1)}{\Gamma(\frac{s}{2}+1)} + \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) \end{aligned} \quad (8)$$

in Eq. (6) lead to the corresponding pole expansion of $Z(s)$. \diamond

Thus, $Z(s)$ has a simple pole at $s = 1$ with the residue

$$\begin{aligned} r_Z(1) &= 2[1 - \log 2\pi] + \gamma + \frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right) - 2 \sum_{\rho} \frac{1}{\rho(1-\rho)}, \\ \gamma + \frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right) &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n(n+\frac{1}{2})}, \end{aligned} \quad (9)$$

at $s = \rho$ with the residue

$$\begin{aligned} r_Z(\rho) &= -2[1 - \log 2\pi] - \frac{2}{\rho-1} + \frac{2}{\rho} + \sum_{n=1}^{\infty} \left(\frac{2}{2n+\rho} - \frac{1}{n} \right), \\ \sum_{n=1}^{\infty} \left(\frac{2}{2n+\rho} - \frac{1}{n} \right) &= -\gamma - \frac{\Gamma'}{\Gamma}\left(1 + \frac{\rho}{2}\right), \end{aligned} \quad (10)$$

at $s = -2n$ for $n = 1, 2, \dots$ and double poles at $s = \rho$, all with coefficients 2, and $s = -2n$ for $n = 1, 2, \dots$.

Lemma 2.2. *The functional equation for $Z(s)$ is given by*

$$\begin{aligned} Z(1-s) &+ \frac{d}{ds} \frac{\Gamma'(s)}{\Gamma(s)} - \frac{(\pi/2)^2}{\cos^2 s\pi/2} = Z(s) + \left(\frac{\Gamma'(s)}{\Gamma(s)} - \log 2\pi - \frac{\pi}{2} \tan \frac{s\pi}{2} \right)^2 \\ &+ 2 \frac{\zeta'(s)}{\zeta(s)} \left[\frac{\Gamma'(s)}{\Gamma(s)} - \log 2\pi - \frac{\pi}{2} \tan \frac{s\pi}{2} \right]. \end{aligned} \quad (11)$$

Proof. Differentiating the functional equation [2],[4],[6] of $\frac{\zeta'(s)}{\zeta(s)}$,

$$- \frac{\zeta'(1-s)}{\zeta(1-s)} = -\log 2\pi - \frac{\pi}{2} \tan \frac{s\pi}{2} + \frac{\Gamma'(s)}{\Gamma(s)} + \frac{\zeta'(s)}{\zeta(s)}, \quad (12)$$

we obtain

$$\frac{\zeta''(1-s)}{\zeta(1-s)} - \left(\frac{\zeta'(1-s)}{\zeta(1-s)} \right)^2 = \frac{d}{ds} \frac{\Gamma'(s)}{\Gamma(s)} + \frac{\zeta''(s)}{\zeta(s)} - \left(\frac{\zeta'(s)}{\zeta(s)} \right)^2 - \frac{(\pi/2)^2}{\cos^2 s\pi/2} \quad (13)$$

and thus the corresponding functional relation for $Z(s)$. \diamond

3 Constraint Generating Dirichlet Series

Definition 3.1. The generating Dirichlet series that implements the constraint in Golomb's identity is defined as

$$Q_D(s) = \sum_{a > [D/2]}^{\infty} \frac{1}{(4a^2 - D^2)^s}, \quad (14)$$

with $[D/2]$ the integer part of $D/2$. It converges absolutely for $\sigma > 1/2$.

Lemma 3.1.

$$Q_D(s) = 4^{-s} \sum_{\nu=0}^{\infty} (-1)^{\nu} \left(\frac{D}{2} \right)^{2\nu} \binom{-s}{\nu} \left[\zeta(2s+2\nu) - \sum_{a \leq [D/2]} a^{-2s-2\nu} \right], \quad (15)$$

where $[D/2]$ is the integer part of $D/2$.

Proof. Using a binomial expansion and interchanging the summations, which is justified by absolute convergence for $\sigma > 1/2$, we obtain Eq. (15). \diamond

Therefore, $Q_D(s)$ has a simple pole at $s = 1/2$ with residue $1/4$ and at $s = \frac{1}{2}(1 - 2\nu)$ with residue $\frac{1}{4}(-D)^\nu \binom{\nu - \frac{1}{2}}{\nu}$ for $\nu = 1, 2, \dots$ and is regular elsewhere.

Definition 3.2. The subtracted constraint Dirichlet series is defined as

$$\begin{aligned} q_D(s) &= Q_D(s) - 2^{-2s} [\zeta(2s) - \sum_{a \leq [D/2]} a^{-2s}] \\ &= \sum_{a > [D/2]} \left(\frac{1}{(4a^2 - D^2)^s} - \frac{1}{(4a^2)^s} \right). \end{aligned} \quad (16)$$

Corollary 3.1. $q_D(s)$ can be resummed as the contour integral

$$q_D(s) = \frac{-1}{2\pi i} \int_C \left(\frac{\Gamma'(2+z)}{\Gamma(2+z)} - \log z \right) [(4z^2 - D^2)^{-s} - (4z^2)^{-s}] dz, \quad \sigma > \frac{1}{2}, \quad (17)$$

where the contour C runs parallel to the imaginary axis from $c - i\infty$ to $c + i\infty$ with $-[D/2] - 1 < c < -[D/2]$, and $[D/2]$ the integral part of $D/2$.

Proof. Adapting a variant of the integral representation of the zeta function due to Kloosterman [4],

$$\begin{aligned} \zeta(2s) - \sum_{a \leq [D/2]} a^{-2s} &= \frac{-1}{2\pi i} \int_{(-[D/2]-1 < c < -[D/2])} \left(\frac{\Gamma'(2+z)}{\Gamma(2+z)} - \log z \right) z^{-2s} dz, \\ \sigma &> 1/2, \end{aligned} \quad (18)$$

where the contour runs parallel to the imaginary axis through the abscissa c . The integral representation can also be obtained by folding the contour to the left, running from $-\infty$ back to $-\infty$ enclosing the point $-[D/2] - 1$ in an anticlockwise sense and applying the residue theorem. \diamond

Lemma 3.2. The Dirichlet series $q_D(s)$ is regular for $\sigma > -1/2$, $\mathcal{O}(1)$ for $|t| \rightarrow \infty$ and, with its general term grouped as $[(2a)^2 - D^2]^{-s} - (2a)^{-2s}$, converges absolutely for $\sigma > -1/2$.

Proof. For a large compared to $|t|$, the convergence is the same as for $t = 0$, i.e. $s = \sigma$, which is a well known property of Dirichlet series. A binomial expansion of the general term of $q_D(\sigma)$,

$$(2a)^{-2\sigma} \left[\left(1 - \frac{D^2}{(2a)^2} \right)^{-\sigma} - 1 \right] = \mathcal{O}(a^{-2\sigma-2}), \quad (19)$$

shows the absolute convergence and regularity for $\sigma > -1/2$. \diamond

The convergence of the integral representation of $q_D(s)$ may be improved by including more terms of Stirling's asymptotic series [5]. This observation leads to

$$q_D(s) = \frac{-1}{2\pi i} \int_C \left(\frac{\Gamma'(2+z)}{\Gamma(2+z)} - \log(z+1) - \frac{1}{2(z+1)} + \frac{1}{12(z+1)^2} \right) \cdot [(4z^2 - D^2)^{-s} - (4z^2)^{-s}] dz. \quad (20)$$

Now we deform the contour to enclose the real axis in the left-hand plane along two rays from the origin with opening angle $\phi/|t|$ around π for fixed $\Im(s) = t \neq 0$ and $-\pi < \phi < \pi$, i.e. $z = -re^{i\phi/|t|}$, $r > 0$ is used to show that the general term is $\mathcal{O}(1)$ for $t \rightarrow \pm\infty$. The conic area bounded by the rays is closed off by a small circular path around $z = -[D/2] - 1$. Stirling's series for the logarithmic derivative of the Gamma function applies outside the cone area guaranteeing convergence of the integral in Eq. (20). Each term in

$$\begin{aligned} & [4z^2 - D^2]^{-s} - (2z)^{-2s} \\ = & [(2r)^2 e^{2i(\pi - \phi/|t|)} - D^2]^{-\sigma - it} - [(2r)^2 e^{2i(\pi - \phi/|t|)}]^{-\sigma - it} \end{aligned} \quad (21)$$

remains bounded, in absolute value, as we let $t \rightarrow \pm\infty$ for $\sigma > -1/2$.

4 Twin Prime Dirichlet Series

The following product formula for absolutely converging Dirichlet series is one of our main tools; it is obtained from summing a well-known formula from which mean values of Dirichlet series are usually derived. We mention it for ease of reference and because details of its proof are needed in Sect. 6.

Lemma 4.1. (*Product formula.*) *Let the Dirichlet series*

$$f(s) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}, \quad g(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s} \quad (22)$$

be single-valued, regular and absolutely convergent for $\sigma > \sigma_a$ and $\sigma > \sigma_b$, respectively. Then the product series

$$P(w) = \sum_{l=1}^{\infty} \frac{a_l b_l}{l^w} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(\sigma + it) g(w - \sigma - it) dt \quad (23)$$

is regular and converges absolutely for $\Re(w) \equiv u > \sigma + \sigma_b, \sigma > \sigma_a + 1$. The limit of the integral exists and is a regular function of the variable $w \equiv u + iv$.

Proof. Substituting the Dirichlet series and using [2],[4]

$$\int_{-T}^T \left(\frac{n}{m}\right)^{it} dt = \begin{cases} 2T, & n = m \\ \frac{2 \sin(T \log n/m)}{\log n/m}, & n \neq m \end{cases}, \quad \sin\left(T \ln \frac{n}{m}\right) = \mathcal{O}(1), \quad T \rightarrow \infty, \quad (24)$$

we obtain the first term on the rhs of

$$\begin{aligned} & \sum_{m,n=1}^{\infty} \frac{a_m b_n}{n^w} \left(\frac{n}{m}\right)^{\sigma} \frac{1}{2T} \int_{-T}^T \left(\frac{n}{m}\right)^{it} dt \\ &= \sum_{n=1}^{\infty} \frac{a_n b_n}{n^w} + \lim_{T \rightarrow \infty} \sum_{1 \leq m \neq n} \frac{a_m b_n}{n^w} \left(\frac{n}{m}\right)^{\sigma} \frac{\sin(T \log n/m)}{T \log n/m} = \sum_{n=1}^{\infty} \frac{a_n b_n}{n^w}, \end{aligned} \quad (25)$$

which converges absolutely because $u \geq \sigma_a + \sigma_b + \epsilon$.

To deal with $\sum_{m \neq n}$ we split the summation $m \neq n$ into the ranges $n \leq m/2, m/2 < n < 2m, n \geq 2m$. Since $|\log \frac{n}{m}| \geq 1/\log 2$ for $n \leq m/2$ we obtain the estimate

$$\left| \sum_{n \leq m/2} \frac{a_m b_n}{n^u} \left(\frac{n}{m}\right)^{\sigma} \frac{\sin(T \log n/m)}{\log n/m} \right| \leq \frac{1}{\log 2} \sum_{n \leq m/2} \frac{|a_m b_n|}{n^{u-\sigma} m^{\sigma}} = \mathcal{O}(1), \quad (26)$$

provided $u - \sigma > \sigma_b$ and $\sigma > \sigma_a$. For $n \geq 2m$, we get the same estimate. For $m/2 < n < 2m$, we split the range $\frac{m}{2} < n < 2m$ into $\frac{m}{2} < n < m$ and $m < n < 2m$ with $n \neq m$ and use

$$\frac{1}{\log(1 - \frac{1}{n})} = n + \mathcal{O}(1), \quad \frac{1}{\log(1 - \frac{2}{n})} = \frac{n}{2} + \mathcal{O}(1), \dots, \quad n \sum_{j=1}^{n-1} \frac{1}{j} = \mathcal{O}(n \log n) \quad (27)$$

to obtain the estimate

$$\sum_{\frac{m}{2} < n < m} \frac{1}{|\log \frac{n}{m}|} = \mathcal{O}(m \log m), \quad (28)$$

and the same estimate for the range $m < n < 2m$. Putting all this together, we find for $\sigma > \sigma_a + 1, u - \sigma > \sigma_b$ that

$$\begin{aligned} \left| \sum_{m/2 < n < 2m} \frac{a_m b_n}{n^u} \left(\frac{n}{m}\right)^{\sigma} \frac{\sin(T \log n/m)}{\log n/m} \right| &= \mathcal{O} \left(\sum_{m/2 < n < 2m} \frac{|a_m b_n| \log m}{n^{u-\sigma} m^{\sigma-1}} \right) \\ &= \mathcal{O}(1). \end{aligned} \quad (29)$$

Using termwise differentiation with respect to the variable w in the integrals in conjunction with the estimate $\log n = \mathcal{O}(n^\epsilon)$ shows the absolute convergence of the termwise differentiated product series and the regularity of the product series. \diamond

As a first step toward constructing the twin prime Dirichlet series we apply the product formula to $Z(s)q_D(s-w)$.

Theorem 4.1. *For odd $D > 0$, $\Re(w) = u > \sigma + \frac{3}{2}$, $\sigma > 1$,*

$$\sum_{a>[D/2]}^{\infty} \frac{2\Lambda(2a-D)\Lambda(2a+D)}{(4a^2-D^2)^w} = A(w) + \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T Z(\sigma+it)q_D(w-\sigma-it)dt, \quad (30)$$

$$A(w) \equiv \sum_{n>[D/2]}^{\infty} \frac{\sum_{d|(2n)^2} \mu(d) \log^2 d}{(2n)^{2w}}, \quad (31)$$

where the limit of the integral on the rhs is a regular function for $u > 5/2$ and the asymptotic series $A(w)$ converges for $u > 1/2$.

Proof. Since $Z(s)$ converges absolutely for $\sigma > 1$, the product $Z(s)2^{-2w-2s}[\zeta(2w-2s) - \sum_{a \leq [D/2]} a^{-2w+2s}]$ converges absolutely for $u - \sigma > 1/2$, i.e. $u > 3/2$. Lemma 4.1 implies that the integral in Eq. (4) is $\mathcal{O}(T)$ and the limit $T \rightarrow \infty$ exists representing a regular function for $\sigma > 5/2$. The integral over $Z(s)2^{-2(w-s)}[\zeta(2(w-s)) - \sum_{a \leq [D/2]} a^{-2w+2s}]$ in Eq. (4) is $\mathcal{O}(T)$. We apply the product formula and obtain the first term on the rhs of Eq. (4) for $u > 5/2$. \diamond

5 The prime pair generating and asymptotic Dirichlet series

In order to analyze the generating Dirichlet series $Z(s)$ introduced in Section 2 and find the Dirichlet series on the rhs of Eqs. (55), (31) in Theor. 4.1, which we call *asymptotic Dirichlet series*, we evaluate the arithmetic functions in their numerators.

Proposition 5.1. *Let $n = \prod_{i=1}^k p_i^{\nu_i}$, $\nu_i \geq 1$, $P = \prod_{i=1}^k p_i$ be prime number decompositions. Then*

$$\sum_{d|n} \mu(d) \log^2 d = \sum_{d|P} \mu(d) \log^2 d. \quad (32)$$

If $n = p$ is prime then

$$\sum_{d|p} \mu(d) \log^2 d = -\log^2 p. \quad (33)$$

If $n = p_i p_j$ for prime numbers $p_i \neq p_j$ then

$$\sum_{d|p_i p_j} \mu(d) \log^2 d = 2 \log p_i \log p_j. \quad (34)$$

If $k \geq 3$ in $P = \prod_{i=1}^k p_i$ for different prime numbers p_i then

$$\sum_{d|P} \mu(d) \log^2 d = 0. \quad (35)$$

Proof. Eq. (32) is obvious from the properties of the Möbius function, as is Eq. (33). Eq. (34) follows from

$$\sum_{d|p_i p_j} \mu(d) \log^2 d = -\log^2 p_i - \log^2 p_j + \log^2 p_i p_j = 2 \log p_i \log p_j. \quad (36)$$

Eq. (35) for $k = 3$ follows from expanding $\log^2 p_i p_j = (\log p_i + \log p_j)^2$ etc.

$$\begin{aligned} \sum_{d|p_i p_j p_l} \mu(d) \log^2 d &= -\log^2 p_i - \log^2 p_j - \log^2 p_l + \log^2 p_i p_j + \log^2 p_i p_l \\ &+ \log^2 p_j p_l - \log^2 p_i p_j p_l = 2 \log p_i \log p_j + 2 \log p_i \log p_l + \log^2 p_i + \log^2 p_j \\ &+ \log^2 p_l + 2 \log p_j \log p_l - \log^2 p_i - \log^2 p_j - \log^2 p_l \\ &- 2 \log p_i \log p_j - 2 \log p_i \log p_l - 2 \log p_j \log p_l = 0. \end{aligned} \quad (37)$$

For the general case k , Eq. (35) is proved by induction. Assuming its validity for k we can show that for $k + 1$:

$$\begin{aligned} 0 &= -\log^2 p_{k+1} + \log^2 p_1 p_{k+1} + \cdots + \log^2 p_k p_{k+1} \\ &- \log^2 p_1 p_2 p_{k+1} - \cdots - \log^2 p_{k-1} p_k p_{k+1} + \log^2 p_1 p_2 p_3 p_{k+1} + \cdots \\ &+ \log^2 p_{k-2} p_{k-1} p_k p_{k+1} \pm \cdots + (-1)^{k+1} \log^2 p_1 p_2 \cdots p_{k+1}. \end{aligned} \quad (38)$$

This follows from verifying that the coefficients of $\log^2 p_i$ are zero, as are those of $2 \log p_i p_j$. In essence, Eqs. (34,35) comprise Golomb's identity (1) for $\Lambda(n), \log^2 n$. \diamond

A comment on the constraint $(2a - D, 2a + D) = 1$ of Golomb's formula is in order. If $d|2D$ and $d|2a - D$ then $d|2a + D$; if $d|2a - D$, but d is not a divisor

of $2D$, then d is not a divisor of $2a + D$. Thus, for odd D , if $d|2a - D$ then d does not divide $2a + D$, except for a **finite** number of divisors of D . Likewise, if $d|3(2a - 1) - D$ and d does not divide $2D$, then d does not divide $3(2a - 1) + D$. If $a = b\delta$ with $\delta|D$ then, for odd D , $2a - D = \delta(2b - \frac{D}{\delta})$, $2a + D = \delta(2b + \frac{D}{\delta})$ and $\sum_{d|\delta^2(4b^2 - D^2/\delta^2)} \mu(d) \log^2 d = 0$ follows from Lemma 5.1. For even D , if $2a - 1 = b\delta$, $\delta|D$ with $\delta \neq 2$ then $9(2a - 1)^2 - D^2 = \delta^2(9b^2 - \frac{D^2}{\delta^2})$ and $\sum_{d|\delta^2(9b^2 - D^2/\delta^2)} \mu(d) \log^2 d = 0$ follows again from Lemma 5.1. Thus, despite $(2a - D, 2a + D) \neq 1$, $(3(2a - 1) - D, 3(2a - 1) + D) \neq 1$, Golomb's identity is trivially satisfied for such values of a .

Lemma 5.1. *The coefficient of the asymptotic Dirichlet series $A(w)$ of Theorem 4.1 is given by*

$$\sum_{d|4a^2} \mu(d) \log^2 d = \begin{cases} 2 \log 2 \log p, & a = 2^i p^j, i \geq 0, j \geq 1 \\ -\log^2 2, & a = 2^i, i \geq 0 \\ 0, & a = 2^j \prod_{i=1}^k p_i^{\nu_i}, p_i \neq 2, k \geq 2, \end{cases} \quad (39)$$

where p, p_i are prime numbers $\neq 2$.

Proof. This follows from Prop. 5.1. Using the prime number decomposition of $a = \prod_{j=1}^k p_j^{\nu_j}$, $P \equiv \prod_{j=1}^k p_j$ in conjunction with

$$\sum_{d|4a^2} \mu(d) \log^2 d = \sum_{d|2P} \mu(d) \log^2 d, \quad (40)$$

the first and third lines of Eq. (39) are immediate consequences of Golomb's identity. The second line is a special case of $\sum_{d|p} \mu(d) \log^2 d = -\log^2 p$, where p is a prime number. \diamond

Theorem 5.1. *For odd $D > 0$, $A(w)$ of Eq. (31) in Theor. 4.1 becomes*

$$\begin{aligned} A(w) = & -2 \log 2 \left[(2^{2w} - 1)^{-1} \left(\frac{\zeta'}{\zeta}(2w) + \frac{\log 2}{2^{2w} - 1} \right) \right. \\ & \left. + \sum_{2^j p^k \leq [D/2], p > 2} \frac{\log p}{2^{2jw} p^{2kw}} \right] - \log^2 2 \left[\frac{1}{2^{2w} - 1} - \sum_{1 \leq 2^j \leq [D/2]} \frac{1}{2^{2jw}} \right]. \end{aligned} \quad (41)$$

Thus, $A(w)$ has a simple pole at $w = 1/2$ with the positive residue $\log 2$.

Proof. Summing the Dirichlet series $A(w)$ using Lemma 5.1 yields

$$\begin{aligned} A(w) = & -\log^2 2 \sum_{2^j > [D/2]} \frac{1}{2^{(j+1)2w}} + 2 \log 2 \sum_{2^j p^k > [D/2], p > 2} \frac{\log p}{2^{2jw} p^{2kw}} \\ = & \log^2 2 \left[-\frac{1}{2^{2w} - 1} + \sum_{1 \leq 2^j \leq [D/2]} \frac{1}{2^{2jw}} \right] + 2 \log 2 \sum_{2^j p^k > [D/2], p > 2} \frac{\log p}{2^{2jw} p^{2kw}}, \end{aligned} \quad (42)$$

from which Eq. (41) and the residue follow. \diamond

6 Limit of the Integral

In order to study the integral of Theor. 4.1 we now evaluate the contour integral

$$\mathcal{I}(w) \equiv \frac{1}{2\pi i} \int_C Z(s) q_D(w-s) ds = I_1 + I_2 + I_3 + I_4, \quad (43)$$

where the contour C is a rectangle with the vertices $1+\epsilon-iT$, $1+\epsilon+iT$, $-\epsilon+iT$, $-\epsilon-iT$ and $T \neq \gamma$, $\rho = \beta + i\gamma$ the general root of $\zeta(s)$ in the critical strip, I_1 is the integral from $1+\epsilon-iT$ to $1+\epsilon+iT$, I_2 from $1+\epsilon+iT$ to $-\epsilon+iT$, I_3 from $-\epsilon+iT$ to $-\epsilon-iT$ and I_4 from $-\epsilon-iT$ to $1+\epsilon-iT$, $\epsilon > 0$. The limit $\lim_{T \rightarrow \infty} \frac{\pi}{T} I_1$ of the integral I_1 in Eq. (55) of Theorem 4.1 needs to be analyzed.

Lemma 6.1. *For $u > 5/2$*

$$\begin{aligned} \mathcal{I}(w) &= q_D(w-1) \left[2(1 - \log 2\pi) + \sum_{n=1}^{\infty} \frac{1}{2n(n + \frac{1}{2})} - 2 \sum_{\rho, |\gamma| < T} \frac{1}{\rho(1-\rho)} \right] \\ &+ \sum_{\rho, |\gamma| < T} \left\{ q_D(w-\rho) \left[2(\log 2\pi - 1) - \frac{2}{\rho-1} + \frac{2}{\rho} \right. \right. \\ &+ \left. \left. \sum_{n=1}^{\infty} \left(\frac{2}{2n+\rho} - \frac{1}{n} \right) \right] - 2q'_D(w-\rho) \right\}. \end{aligned} \quad (44)$$

Proof. Eq. (44) follows from using Eq. (7) and the residue theorem taking into account the poles at $s = 1, \rho$ inside the rectangular contour.

For the evaluation of I_2, I_4 we need bounds for $Z(s), q_D(w-s)$ for $s = \sigma \pm iT$.

Lemma 6.2. $I_2 = \mathcal{O}(\log^3 T), I_4 = \mathcal{O}(\log^3 T)$ for $u > 5/2$ and all sufficiently large and appropriately chosen T .

Proof. We know [6] that for sufficiently large $|t|$ and $-1 < \sigma < 2$

$$-\frac{\zeta'(s)}{\zeta(s)} = - \sum_{\rho, |t-\gamma| < 1} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) + \mathcal{O}(\log |t|). \quad (45)$$

Choosing $t = \Im(s), T = |t|$ sufficiently large and appropriately we can arrange that $|t-\gamma| \geq \frac{1}{\log T}$, when $|\gamma-t| < 1$. There are at most $\mathcal{O}(\log T)$

roots ρ with $|\gamma - t| < 1$. As a result[6],

$$\sum_{\rho} \frac{1}{|s - \rho|^2} = \sum_{\rho, |\gamma - T| \geq 1} \frac{1}{|s - \rho|^2} + \sum_{\rho, |\gamma - T| < 1} \frac{1}{|s - \rho|^2} = \mathcal{O}(\log T) + \mathcal{O}(\log^3 T). \quad (46)$$

Hence $|Z(s)| = \mathcal{O}(\log^3 T)$ for I_2 and the same bound holds for I_4 and $-\epsilon \leq \sigma \leq 1 + \epsilon$.

On all four sides of the rectangular path $-\epsilon \leq \sigma \leq 1 + \epsilon$, $\Re(w - s) > -\frac{3}{2}$ for $u > 3/2$ the Dirichlet series $q_D(w - s)$ remains bounded for $u - \sigma > 1/2$, in absolute magnitude, for $|t| \rightarrow \infty$ according to Lemma 3.2 . \diamond

Lemma 6.3. For $u > 5/2$, $I_3 = \mathcal{O}(\log^2 T)$.

Proof. For I_3 we will use the functional equation (11) of $Z(s)$ to evaluate the first term on the rhs of

$$\begin{aligned} I_3 &= -\frac{1}{2\pi i} \int_{-\epsilon - iT}^{-\epsilon + iT} q_D(w - s) \left\{ Z(1 - s) + \frac{d}{ds} \frac{\Gamma'(s)}{\Gamma(s)} - \frac{(\pi/2)^2}{\cos^2 s\pi/2} \right. \\ &\quad - \left. \left(\frac{\Gamma'(s)}{\Gamma(s)} - \log 2\pi - \frac{\pi}{2} \tan \frac{s\pi}{2} \right)^2 \right. \\ &\quad \left. - 2 \frac{\zeta'(s)}{\zeta(s)} \left[\frac{\Gamma'(s)}{\Gamma(s)} - \log 2\pi - \frac{\pi}{2} \tan \frac{s\pi}{2} \right] \right\} ds, \end{aligned} \quad (47)$$

$q_D(w - s)Z(1 - s)$, by substituting the absolutely converging Dirichlet series. This yields a double series $\sum_{m,n}$ as for I_1 in Lemma 4.1 involving the integral $\int_{-T}^T (m[(2n)^2 - D^2])^{it} dt$. Thus, there is no contribution for $4n^2 - D^2 = m$ that would be proportional to T . As earlier, $\sum_{m \neq (2n)^2 - D^2}$ involves $\sin(T \log[(2n)^2 - D^2]m)$ and is $\mathcal{O}(1)$ for $u + \epsilon > 3/2$. The same applies to the double sums involving $(2n)^2$ and $4n^2 - 2D^2$ instead of $(2n)^2 - D^2$.

To deal with the term involving $-\frac{\zeta'(s)}{\zeta(s)} 2(\gamma + \log 2\pi) q_D(w - s)$ we use the functional equation of Lemma 2.2 and then substitute the absolutely converging Dirichlet series for $q_D(w - s)$ and $-\frac{\zeta'(1-s)}{\zeta(1-s)}$ at $\sigma = -\epsilon$. Again, each double series is obtained as for I_1 in Lemma 4.1 involving the integral $\int_{-T}^T (m[(2n)^2 - D^2])^{it} dt$, $\int_{-T}^T (m(2n)^2)^{it} dt$, or $\int_{-T}^T [m(4n^2 - 2D^2)]^{it} dt$, respectively. Again, there is no contribution for $(2n)^2 - D^2 = m$, $(2n)^2 - 2D^2 = m$, or $(2n)^2 = m$, that would be proportional to T . As earlier, $\sum_{m \neq (2n)^2 - D^2}$, $\sum_{m \neq (2n)^2 - 2D^2}$, or $\sum_{m \neq (2n)^2}$ lead to the bound $\mathcal{O}(1)$.

Putting all this together, we obtain

$$I_3 = -\frac{1}{2\pi i} \int_{-\epsilon - iT}^{-\epsilon + iT} q_D(w - s)$$

$$\begin{aligned}
& \cdot \left\{ \frac{d}{ds} \frac{\Gamma'(s)}{\Gamma(s)} - \frac{(\pi/2)^2}{\cos^2 s\pi/2} - \left(\frac{\Gamma'(s)}{\Gamma(s)} - \log 2\pi - \frac{\pi}{2} \tan \frac{s\pi}{2} \right)^2 \right. \\
& + \left[-2 \log 2\pi - \pi \tan \frac{s\pi}{2} + 2 \frac{\Gamma'(s)}{\Gamma(s)} \right] \left[\frac{\Gamma'(s)}{\Gamma(s)} - \log 2\pi - \frac{\pi}{2} \tan \frac{s\pi}{2} \right] \\
& + \left. 2 \frac{\zeta'(1-s)}{\zeta(1-s)} \left[\frac{\Gamma'(s)}{\Gamma(s)} + \gamma - \frac{\pi}{2} \tan \frac{s\pi}{2} \right] \right\} ds + \mathcal{O}(1). \tag{48}
\end{aligned}$$

The term $\frac{1}{2\pi i} \int_{-\epsilon-iT}^{-\epsilon+iT} q_D(w-s) \frac{(\pi/2)^2}{\cos^2 s\pi/2} ds = \mathcal{O}(1)$ because $(\cos^2 s\pi/2)^{-2} = \mathcal{O}(e^{-|t|\pi})$ for $t \rightarrow \pm\infty$.

The terms involving $|\tan \frac{s\pi}{2}| \rightarrow 1$ for $t \rightarrow \pm\infty$, so $|\frac{d}{ds} \tan \frac{s\pi}{2}| = \mathcal{O}(e^{-|t|\pi/2})$. Using integration by parts, we get the following estimate for this term from the Dirichlet series for $q_D(w-s)$

$$\begin{aligned}
& \int_{-\epsilon-iT}^{-\epsilon+iT} \tan \frac{s\pi}{2} \sum_{m=2}^{\infty} [(4m^2 - D^2)^{s-w} - (4m^2)^{s-w}] ds \\
& = \sum_{m=2}^{\infty} [(4m^2 - D^2)^{s-w} (\log(4m^2 - D^2))^{-1} \\
& - (4m^2)^{s-w} (2 \log(4m^2))^{-1}] \tan \frac{s\pi}{2} \Big|_{s=-\epsilon-iT}^{s=-\epsilon+iT} \\
& - \int_{-\epsilon-iT}^{-\epsilon+iT} \sum_{m=2}^{\infty} [(2m)^2 - D^2]^{s-w} [\log((4m^2 - D^2))^{-1} \\
& - (4m^2)^{s-w} (2 \log(4m^2))^{-1} \\
& - (4m^2 - 2D^2)^{s-w} (2 \log(4m^2 - 2D^2))^{-1}] \frac{d}{ds} \tan \frac{s\pi}{2} ds,
\end{aligned}$$

where the integrated term is $\mathcal{O}(1)$ because the Dirichlet series is absolutely convergent, $\tan \frac{s\pi}{2} = \mathcal{O}(1)$ and the integral is $\mathcal{O}(1)$. Terms involving $\tan^2 \frac{s\pi}{2}$, $\tan \frac{s\pi}{2} \frac{\Gamma'(s)}{\Gamma(s)}$ are handled in the same way.

The terms involving $q_D(w-s) \frac{\Gamma'(s)}{\Gamma(s)}$ are also treated by integration by parts where the integrated term is $\mathcal{O}(\log T)$ and the integral is integrated by parts leading to $\frac{d}{ds} \frac{\Gamma'(s)}{\Gamma(s)} = \mathcal{O}(T^{-1})$ at $s = -\epsilon \pm iT$. Hence the remaining integral is $\mathcal{O}(\log T)$. The term involving $q_D(w-s) \frac{d}{ds} \frac{\Gamma'(s)}{\Gamma(s)}$ is treated by integration by parts leading to the same estimate. The term $q_D(w-s) \left(\frac{\Gamma'(s)}{\Gamma(s)} \right)^2$ is also treated by integrating by parts leading to an $\mathcal{O}(\log^2 T)$ estimate.

The product $q_D(w-s)\frac{\zeta'(1-s)}{\zeta(1-s)}$ is another absolutely converging Dirichlet series without constant term, i.e. unity. All these terms in Eq. (48) are also treated by integration by parts leading to the estimate $\mathcal{O}(\log T)$. Putting all this together proves Lemma 6.3. \diamond

Theorem 6.1. *For odd $D > 0, u > 5/2$,*

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{\pi \mathcal{I}(w)}{T} &= \lim_{T \rightarrow \infty} \frac{\pi I_1}{T} = \lim_{T \rightarrow \infty} \frac{\pi}{T} \sum_{\rho, |\gamma| < T} \left\{ q_D(w - \rho) \cdot \right. \\ &\quad \left[2(\log 2\pi - 1) - \frac{2}{\rho - 1} + \frac{2}{\rho} + \sum_{n=1}^{\infty} \left(\frac{2}{2n + \rho} - \frac{1}{n} \right) \right] - 2q'_D(w - \rho) \right\}, \\ \lim_{T \rightarrow \infty} \frac{\pi I_1}{T} &= \sum_{a > [D/2]} \frac{2\Lambda(2a - D)\Lambda(2a + D)}{(4a^2 - D^2)^w}. \end{aligned} \quad (49)$$

Proof. We apply the limit $T \rightarrow \infty$ on $\mathcal{I}(w)$ to Eqs. (43),(44) using the bounds on I_2, I_3, I_4 from Lemmas 6.2, 6.3 to obtain Eq. (49) because the first line of Eq. (44) and the first term of the second line drop out. The last line follows from Theor. 4.1 and Lemma 4.1. \diamond

This is our main result. Due to the limit $T \rightarrow \infty$ in Theor. 6.1 the twin prime distributions depend on the asymptotic properties of the roots of the Riemann zeta function. This feature contrasts with the analytic proof of the prime number theorem, where the remainder term is linked to the roots of the Riemann zeta function producing the staircase-like corrections of the smooth asymptotic limit from **all** zeta function roots, whereas the leading asymptotic term has nothing to do with them, originating from the simple pole of the Riemann zeta function.

7 Twin Primes For Even D

Definition 7.1. The corresponding constraint generating Dirichlet series is defined as

$$Q_D(s) = \sum_{a > [D/6]} \frac{1}{[3^2(2a - 1)^2 - D^2]^s}, \quad \sigma > 1/2, \quad (50)$$

with $[D/6]$ the integer part of $D/6$.

Lemma 7.1. *The expansion corresponding to Lemma 3.1 becomes*

$$Q_D(s) = \sum_{\nu=0}^{\infty} (-1)^{\nu} \left(\frac{D}{3} \right)^{2\nu} 3^{-2s} \binom{-s}{\nu} \left[\left(1 - \frac{1}{2^{2s+2\nu}} \right) \zeta(2s + 2\nu) \right]$$

$$- \sum_{a \leq [D/6]} \frac{1}{(2a-1)^{2s+2\nu}} \Big]. \quad (51)$$

Definition 7.2. The subtracted constraint Dirichlet series is defined as

$$q_D(s) = Q_D(s) - 3^{-2s} \left[(1 - 2^{-2s}) \zeta(2s) - \sum_{a \leq [D/6]} (2a-1)^{-2s} \right]. \quad (52)$$

Equation (20) of Lemma 3.2 for this case becomes

$$\begin{aligned} q(s) &= \frac{-1}{2\pi i} \int_C \left(\frac{\Gamma'(\frac{z-1}{2})}{\Gamma(\frac{z-1}{2})} - \log\left(\frac{z-3}{2}\right) - \frac{1}{z-3} + \frac{1}{3(z-3)^2} \right) \\ &\cdot \left[(9z^2 - D^2)^{-s} - (3z)^{-2s} \right] dz, \end{aligned} \quad (53)$$

with $-2[D/6] < c < -2[D/6] + 1$. Lemma 3.2 is valid for this case with the general term grouped as

$$\frac{1}{[3^2(2a-1)^2 - D^2]^s} - 3^{-2s}(2a)^{-2s}. \quad (54)$$

Theorem 7.1. For even $D > 0$, $\Re(w) = u > \sigma + \frac{3}{2}, \sigma > 1$,

$$\begin{aligned} &\sum_{a > [D/6]}^{\infty} \frac{2\Lambda(3(2a-1) - D)\Lambda(3(2a-1) + D)}{[9(2a-1)^2 - D^2]^w} = A(w) \\ &+ \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T Z(\sigma + it) q(w - \sigma - it) dt, \end{aligned} \quad (55)$$

with

$$A(w) \equiv 3^{-2w} \sum_{n > [D/6]}^{\infty} \frac{\sum_{d|3^2(2n-1)^2} \mu(d) \log^2 d}{(2n-1)^{2w}}. \quad (56)$$

This is Theor. 4.1 for even D .

Lemma 7.2. The general coefficient of the asymptotic Dirichlet series $A(w)$ is given by

$$\sum_{d|9(2a-1)^2} = \begin{cases} -\ln^2 3, & 2a-1 = 3^i \\ 2\ln 3 \log p, & 2a-1 = p^i, \ p \neq 3 \\ 0, & \text{else.} \end{cases} \quad (57)$$

Theorem 7.2. *For even $D > 0$, $A(w)$ of Eq. (31) in Theor. 4.1 becomes*

$$\begin{aligned}
A(w) &= -\log^2 3 \sum_{3^j > [D/6]} 3^{-2jw} + 2 \log 3 \sum_{3^{2jw} p^{2kw} > [D/6], p \neq 3} \frac{\log p}{3^{2jw} p^{2kw}} \\
&= -\log^2 3 \left[\frac{1}{3^{2w} - 1} - \sum_{1 \leq 3^j \leq [D/6]} \frac{1}{3^{2jw}} \right] \\
&\quad - 2 \log 3 \left[(3^{2w} - 1)^{-1} \left(\frac{\zeta'}{\zeta}(2w) + \frac{\log 3}{3^{2w} - 1} \right) \right. \\
&\quad \left. + \sum_{3^j p^k \leq [D/6], p \neq 3} \frac{\log p}{3^{2jw} p^{2kw}} \right]. \tag{58}
\end{aligned}$$

Thus, $A(w)$ has a simple pole at $w = 1/2$ with the residue $\log 3$.

Theor. 6.1 remains valid, except for replacing the twin Dirichlet series in Eq. (49) by the lhs of Eq. (55) in Theor. 7.1.

8 Discussion and Conclusion

The twin prime formulas link the twin primes with the roots of the Riemann zeta function, which may be viewed as a step in Riemann's program linking prime numbers to these roots. Their dependence on the limit $T \rightarrow \infty$ demonstrates that the twin prime distributions depend on the asymptotic properties of these roots. This qualitative feature differs fundamentally from the prime number theorem for ordinary prime numbers where all roots contribute to the remainder term.

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